**Excitations from GL Free Energy**

**Ginzburg-Landau Free Energy (Functional)**

Since our Mean Field Theory approach necessarily involves thermal averaging – well all of our GF’s involve thermal averaging, seems like we could use the free energy itself to get the energies. So let’s do that. We’ll look to calculate the Landau free energy L = -kBTln(Ξ) (recall from Stat Mech). So we start with the H:



Then recall from the Stat Mech/G, Z as Path Integral file, where we said we could write Ξ as:



where,



and K = H – μN. It then stands to reason that we may equivalently say (or could make a change of variables if we have to):



where,



The quartic term prevents us from performing the integral. But as is standard, we can use a Hubbard-Stratonovich identity thing to turn the quartic term into a quadratic term. For comparison, recall from the Path Integrals file, the following identity involving integrals over complex variables:



(and in what follows we can ignore the det(A/2π) factor as it is independent of any operator terms – it’s just a number) So at first I tried,



which is true (ignoring det(A/2π) again), but not what we want. So better is:



So we don’t want the Δ(τ) to be k-dependent, so no Δk(τ). And last comment: to get the ’s and c’s in the proper order above, we had to transpose some of them, which we can do at cost of minus signs of course. Filling this in,



We can do the c integration now, but need to diagonalize the c quadratic form. We’ll write as a matrix. Let’s introduce, as we have before in previous files, the Nambu spinors.



And just to save space, let’s also absorb the constraint, |ξk| < ωD, on the summation into a theta step function, θ(|ξk| < ωD), which is 1 if the constraint is satisfied and 0 otherwise. And then let’s define,



and write,



The bottom right element of the matrix requires IBP and some anticommutation of the c’s to put in the given order. And the Δ off-diagonal terms also require anticommutation of the the c’s to put things in proper order (the Δ’s are complex numbers though themselves). Now we can do the c integration. Recall, the generic formula (see Path Integrals folder),



And this time we cannot ignore the detA because it contains important operators, and variables we have yet to integrate over. So we have, making a change of variables c → α (Jacobian of transformation would be a simple number – and so not important):



where we define/recall,



To be a little more precise by what is meant by det(∂τ2 – k2), we’re saying (see Path Integrals folder for more on this) that:



where the operator/matrix M has components defined via:



where |τ> is a ‘time’ eigenstate, analogous to |x> in Quantum Mechanics Hilbert Space notation stuff. To evaluate this determinant, we’d need the eigenvalues of the operator. So we need to work out,



which in ‘time’ space, would read:



Well, k is time-dependent too, due to its dependence on Δ(τ). So that makes working out the eigenfunctions more difficult. But if we go back to when we introduced the Δ guys in the first place, we can see that letting it be a function of time was actually unnecessary. So let’s take Δ(τ) to be independent of τ. Then the eigenfunctions are:



where n1/2 = half-integers (because as we recall, Fermions must obey anti-periodic boundary conditions over interval (0,β), while Bosons obey periodic boundary conditions over (0,β)). So we have:



So now,



Seems we could say,



where we ignore the constant Σkn(πi). But if we do this, then we don’t get the correct result for the gap function. So I guess we’ll stick with:



By the way, the D[Δ\*Δ] is more like dΔ\*dΔ now. Now let’s do the frequency sum. We already did a simpler version of this frequency sum in the Stat Mech/GF,Z – Path Integral file. So let,



and use the identity (see Stat Mech/Math Appendix),



(ε = 1 for Fermions) and as in the aforementioned file, we’ll ignore the convergence issues at |z| → ∞. Now the f(z) function has no poles per se’, but it does have a branch point at z = k, and so we’ll make a branch cut along Re(z) > k. Contour looks like this, but infinitely large.

Chart, diagram

Description automatically generated with medium confidence

Then we have, using the principle value of the ln(z) function [ln(x+i0+) = ln|x|, and ln(x – i0+) = ln|x| + 2πi0+]:



Now gotta do the integral,



So we have:



Doing the same for the other ln term in the exponent will give us:



And altogether,



Well this is exact so far, and if we could do the integral then we’d have an exact solution to the BCS Hamiltonian. Alas, this doesn’t seem to be forthcoming due to the complicated Δ dependence in those ln’s. But we can do a mean field/saddle point approximation instead. And then if we wanted, we could study fluctuations about the mean field (see Cond Matt/Metals/Electron Interactions/Exchange/Thermal Properties folder for examples of that). So anyway. The mean field contribution is obtained by finding the saddle point equation for Δ. So we take a derivative of S w/r to Δ and set equal to zero. And remember Δ doesn’t really depend on k – it’s just a notation to incorporate the θ function constraint,



So we have:



So then,



This is the same gap equation as we’ve found twice before via other means.